ARTICLE XI.

Expansion of $F(x+h)$. *By Pike Powers, of the University of Virginia. Read April* 2, 1841.

A FUNCTION may be regarded as the general expression of a series of numbers which vary according to some given law; the place or number of the term being denoted by x, and its value by Fx. The series may always be represented wholly or in part by a curve whose ordinates correspond to the different values of Fx, and its abscissas to those of x.

In any function certain values may always be assigned to x, between which the difference of any two consecutive values of Fx will not be *infinitely* greater than the difference of the corresponding values of x.

The only functions which the writer can conceive of as not subject to the preceding remark are, 1. Such as undergo, incessantly, abrupt changes from increase to decrease, or the reverse: 2. Those which, while they vary in the same sense through a finite interval, yet undergo always an infinite change for a finite change in x. It seems obvious, however, that if such functions can be analytically expressed, they cannot admit of Taylor's theorem. (See note.)

Supposing x to be confined within the limits referred to above, we have

$$\frac{F(x+h)-Fx}{h}=\phi(x.h); \tag{1}$$

$\phi(x.h)$ reducing to a finite quantity $\phi(x.0)$ when $h=0$.

In like manner we may write

$$\frac{\phi(x.h)-\phi(x.0)}{h}=\phi'(x.h),$$

or

$$\phi(x.h)=\phi(x.0)+h.\phi'(x.h),$$

where $h\,\phi'(x.h)=0$ when $h=0$.

Now if $\phi'(x.h)$ should be finite when $h=0$, we should have

$$\phi'(x.h)=\phi'(x.0)+h.\phi''(x.h);$$

and, by continuing this process, and substituting for $\phi''(x.h)$, $\phi'(x.h)$, $\phi(x.h)$ &c., their values, we should get easily the common development of $F(x+h)$.

But let us suppose that $\phi'(x.0)$ is infinite, that is, that as h approaches 0, $\phi'(x.h)$ increases without limit according to a law depending upon the form of the function.

Whatever this law may be, as functions vanishing with h admit of an infinite diversity of form, it seems obvious that there must be some one fh, such that $\frac{\phi''(x.h)}{fh}$ shall increase with the same rate as $\phi'(x.h)$, $\phi''(x.0)$ being finite. We may write, then,

$$h.\ \phi'(x.h) = \frac{h}{fh}.\ \phi''(x.h).$$

As fh is inferior to h in degree, since $h.\ \phi'(x.h)$ vanishes with h, and as the powers of h admit of every shade of magnitude, and diminish towards 0 with every possible degree of rapidity, it appears evident that for very small values of h, fh may be replaced by h^v where $v < 1$. Hence

$$\frac{h}{fh}\phi''(x.h) = h^{1-v}\, Q' = h^{\alpha}\, Q',$$

where $\alpha = 1 - v$, and Q' is so chosen as to agree with $Q''^{v}(x.h)$ for very small values of h. By similar reasoning we have

$$Q' = \phi''(x.0) + h^{\beta}.\ Q''.$$

Putting Q for $\phi(x.h)$ we may write (1) under the form

$$F(x+h) = Fx + h.\ Q.$$

And if we replace $\phi(x.0)$, $\phi'(x.0)$, &c., by P, P', &c., we shall have

$$\begin{aligned} F(x+h) &= Fx + h\ .\ Q, \\ Q &= P + h^{\alpha}.\ Q', \\ Q' &= P' + h^{\beta}.\ Q'', \\ &\ \ \vdots \\ Q''^{n} &= P''^{n} + h^{\sigma}.\ `Q; \end{aligned}$$

$`Q$ being finite when $h = 0$. Multiplying the 2d equation by h, the 3d by $h^{1+\alpha}$, &c., adding, and putting a for $1+\alpha$, b for $1+\alpha+\beta$, &c., we have

$$F(x+h) = Fx + Ph + P'h^{a} + P''h^{b} + \ldots\ldots\ h^{s}.\ Q; \qquad (2)$$

which differs only in the 2d term from the development assumed by Poisson.

The equation

$$F(x+h) = Fx + Ph + h.\ R,$$

which is derived from (1) by a simple transformation, R taking the place of $h.\ \phi'\,(x\,.\,h)$, and consequently vanishing with h, is sufficient to establish all the rules of differentiation.

Observing that in the preceding investigation x was confined within certain limits, while h remained arbitrary, we may replace x by a number r within the limits supposed, and h by $x - r$ which denotes the variable difference between the general and special values of x. Equation (2) will then become

$$u = Fx = Fr + p\,(x - r) + p'\,(x - r)^a + p''\,(x - r)^b + \ldots\ldots M;$$

where $p\,p'\,p''$, &c., denote the values of $P\,P'\,P''$, &c., when $x = r$, and M the value of $h^s.\ `Q$ when $x = r$, and $h = x - r$.

Differentiating successively, and denoting the differential coefficients of M by $M'\,M''$, &c., we have

$$\begin{aligned}
\frac{du}{dx} &= p + ap'\,(x - r)^{a-1} + bp''\,(x - r)^{b-1} + \ldots\ldots\ldots\ldots\ M', \\
\frac{d^2u}{dx^2} &= a\,(a - 1)\,(x - r)^{a-2} + b\,(b - 1)\,p''\,(x - r)^{b-2} + \ldots\ldots\ M'', \quad (3) \\
\frac{d^3u}{dx^3} &= a\,(a - 1)\,(a - 2)\,(x - r)^{a-3} + b\,(b - 1)\,(b - 2)\,(x - r)^{b-3} + M'''.
\end{aligned}$$

We may suppose that each term in these equations is the only one which contains the power of $x - r$ peculiar to it, for if there were several terms containing the same power of $x - r$, they might be united into one.

Observing, now, that the exponents a, b, c, &c., are each greater than unity, and are arranged in ascending order, if we make $x = r$, and suppose $\frac{du}{dx}, \frac{d^2u}{dx^2}$, &c., to remain finite, the first of equations (3) becomes

$$\left(\frac{du}{dx}\right) = p;$$

a result already established, and implied in the process of differentiating.

With regard to the 2d equation we must have

$$a > 2, \text{ or } a < 2, \text{ or } a = 2.$$

If $a > 2$, every term in the 2d member antecedent to M'' will vanish, and if M'' does not vanish, it must either be finite or infinite. But, since $\left(\frac{d^2u}{dx^2}\right)$ is finite, M'' cannot be infinite. If it reduces to a finite quantity A, then M' *

* See Mr. Bonnycastle's paper, pages 245, 246, Vol. VII. of these Transactions.

must contain a term $A\ (x - r)$, and M a term $A\ (x - r)^2$, and it is only necessary to give this term its proper place in the series in order to get the same result which $a = 2$ will furnish. If $a < 2$, the 1st term will be infinite, and cannot be cancelled by any of the terms antecedent to M$''$, since they all contain powers of $x - r$ different from the first; nor by any term in M'', since that term would then contain the same power of $x - r$ with the first, which is contrary to the arrangement of the series. We must have, then, $a = 2$. Therefore $a\ (a - 1) = 1\ .\ 2$,—all the terms after the first vanish, since M'' cannot remain finite for the same reason as in the preceding case,—and we have

$$\left(\frac{d^2 u}{d\, x^2}\right) = 1.\ 2.\ p' \quad \therefore \quad p' = \frac{1}{1\ .\ 2}\left(\frac{d^2 u}{d\, x^2}\right).$$

In the same way we can show that

$$p'' = \frac{1}{1\ .\ 2\ .\ 3}\left(\frac{d^3 u}{d\, x^3}\right), \ \&c.$$

As r is any value of x between the supposed limits, the results obtained are evidently general, and will give all the terms of the series until we reach a coefficient which becomes infinite for $x = r$, and then the remainder of the expansion must be supplied in some other way. Using the notation of Lagrange, we have generally, therefore,

$$F x = F r + F' r\ (x - r) + F'' r\ .\ \frac{(x - r)^2}{1\ .\ 2} + \ .\ .\ .\ .\ .\ .\ (x - r)^s.\ q;$$

which, when $r = 0$, becomes

$$F x = F\ (0) + x.\ F'\ (0) + \frac{x^2}{1\ .\ 2} F''\ (0) + \ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ x^s.\ q'.$$

Equation (2) also becomes

$$F\ (x + h) = F x + h.\ F' x + \frac{h^2}{1\ .\ 2} F'' x + \ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ h^s.\ `Q.$$

The limits within which the value of the last term is found may be determined by the method of Lagrange, and the development will be complete. The reader will find a summary view of this method in the subjoined note, furnishing itself an exact though indirect solution of the problem.

NOTE.

The validity of the reasoning used in the foregoing demonstration, to show the existence of $F'x$ in a finite form in all cases where x is confined within certain limits, will perhaps appear more evident from the following remarks.

Postulate 1. There are no functions which, throughout their whole range of values, change incessantly from increase to decrease as x varies, and that by quantities infinitely greater than the change in x. It is scarcely possible to give the graphic representation of such functions, much less their analytical expression. A line continually returning upon itself thus, , or a spiral whose coils are compressed into almost absolute contact, would be an approximative expression of them. We conclude, then, that in any function Fx, values a and $a+nh$ may be assigned to x, differing by a finite quantity nh, and such that from Fa to $F(a+nh)$, Fx shall constantly increase or constantly decrease.

Postulate 2. There are no functions which, while they undergo a constant increase or decrease through finite intervals of value, yet always receive an infinite change for a finite change in x. And here we again appeal to observation, and the apparent impossibility of exhibiting such functions in either a geometric or algebraic form.

Theorem 1. Now suppose that $\dfrac{F(x+h)-Fx}{h}$ approaches infinity as h approaches 0, for all values of x. Then the following ratios,

$$\frac{F(x+h)-Fx}{h},\ \frac{F(x+2h)-F(x+h)}{h},\ \frac{F(x+3h)-F(x+2h)}{h},\ \ldots\ \frac{F(x+nh)-F[x+(n-1)h]}{h},$$

will all be infinitely great when h is infinitely small.

Let n be taken so great that nh shall be finite, and let x be such that Fx constantly increases or constantly decreases from Fx to $F(x+nh)$. The numerators of the preceding ratios will be all of the same sign; their sum is obviously $F(x+nh)-Fx$; and if P denote the least of these numerators, $nP < F(x+nh)-Fx$. But

$$\frac{P}{h}=\frac{nP}{nh}=\infty;\ \text{hence}\ \frac{F(x+nh)-Fx}{nh}=\infty.$$

But this result is impossible by postulate 2. Hence in any function Fx there must be some values of x, such that

$$\frac{F(x+h)-Fx}{h} = \text{a finite quantity } F'x, \text{ when } h=0.$$

From which we readily derive

$$F(x+h)=Fx+h\,.\,F'x+h\,.\,R,$$

R vanishing with h.

As to the method used by Lagrange to determine the limits of the expansion of $F(x+h)$, it may not be amiss to observe that when the existence of the differential coefficients in a finite form is admitted, this method furnishes in all cases an exact and simple mode of exhibiting the true value of $F(x+h)$. This fact has been most singularly overlooked.

Cauchy apparently, and De Morgan confessedly, have made Lagrange's method the basis of their demonstrations of Taylor's theorem. We will now exhibit the method of Lagrange, after premising that the equation

$$F\,(x + h) = Fx + h\,.\,F'x + h\,.\,R$$

will readily establish the following well known theorem.

Theorem 2. When $F'x$ is positive, Fx and x vary in the same sense, and when negative, in an opposite sense; consequently, if $F\,(0) = 0$, and $F'\,(0)$ be not infinite, Fx will be positive or negative at the same time with x, when $F'x$ is positive.

Let us suppose that for $x = a$, and $x = b$, and for all intermediate values, $F'x$, $F''x \ldots\ldots F^{(n)}x$ are all finite and continuous, and let us replace x by $a + h$, h admitting all values from 0 to $b - a$.

Now let A and B be the greatest and least values of $F'\,(a + h)$: then

$$A - F'\,(a + h) > 0, \text{ and } F'\,(a + h) - B > 0.$$

Hence the primitives of these expressions taken with regard to h, and so as to vanish with h, will likewise be positive. Theorem (2).

$$A\,h - F\,(a + h) + Fa > 0, \text{ and } F\,(a + h) - B\,h - Fa > 0.$$

Next let A' and B' be the greatest and least values of $F''\,(a + h)$: then

$$A' - F''\,(a + h) > 0, \text{ and } F''\,(a + h) - B' > 0.$$

By taking the primitives as before, we have

$$A'h - F'\,(a + h) + F'a > 0, \text{ and } F'\,(a + h) - B'h - F'a > 0;$$

and by taking them again, we have

$$A'\frac{h^2}{2} - F(a + h) + h\,.\,F'a + Fa > 0, \text{ and } F(a + h) - B'\,\frac{h^2}{1\,.\,2} - h\,.\,F'a - Fa > 0,$$

or

$$F\,(a + h) < Fa + h\,.\,F'a + \frac{h^2}{1\,.\,2}\,A', \text{ and } F\,(a + h) > Fa + h\,.\,F'a + \frac{h^2}{1\,.\,2}\,B'.$$

By continuing this process, we shall finally get

$$F(a + h) < Fa + h\,.\,F'a + \frac{h^2}{2}\,.\,F''a + \frac{h^3}{1\,.\,2\,.\,3}\,.\,F'''a + \ldots\ldots \frac{h^n}{1\,.\,2\,.\,3\,.\,.\,n}\,.\,A^{(n)}$$

$$F(a + h) > Fa + h\,.\,F'a + \frac{h^2}{1\,.\,2}\,.\,F''a + \frac{h^3}{1\,.\,2\,.\,3}\,.\,F'''a + \ldots\ldots \frac{h^n}{1\,.\,2\,.\,3\,.\,.\,n}\,.\,B^{(n)}$$

$A^{(n)}$ and $B^{(n)}$ representing the greatest and least values of $F^{(n)}\,(a + h)$.

Hence if $F^n\,(a + h)$ be continuous, and $h = b - a$, there will be some value $F^n\,(a + \theta\,h)$ intermediate between $A^{(n)}$ and $B^{(n)}$ such that

$$F\,(a + h) = Fa + h\,.\,F'a + \frac{h^2}{1\,.\,2}\,.\,F''a + \ldots\ldots\ldots \frac{h^n}{1\,.\,2\,.\,3\,.\,.\,n}\,.\,F^n\,(a + \theta\,h),$$

where $\theta < 1$. This is precisely the expression obtained by Cauchy. It is general, since a is any value of x subject to the conditions stated, and it gives always the exact value of $F\,(x + h)$ when x and h are such that $F^n\,(x + \theta\,h)$ shall be finite, since $\dfrac{h^n}{1\,.\,2\,.\,3\,.\,.\,n}$ will always finally converge.

With regard to the negative values of h, we shall have, by theorem (2),

$$\mathcal{A} - F'(a-h) > 0,\ F'(a-h) - B > 0,$$

$$\mathcal{A}h - F(a-h) + Fa < 0,\ F(a-h) + Bh - Fa < 0,$$

or

$$F(a-h) > Fa - \mathcal{A}h,\ F(a-h) < Fa - Bh;$$

and the reasoning continued as before will lead to a similar result.

It may be observed, in conclusion, that the integrations effected above are perfectly allowable, since the equation

$$F(x+h) = Fx + h \,.\, F'x + h \,.\, R$$

is sufficient for all purposes of differentiation and integration. And it is immaterial whether any other primitives than those obtained exist, since we are not seeking *the only expansion* of $F(x+h)$, but *one true expansion* of it. (See Calc. des Fonctions, Leçon 9me.)